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Approximate Natural Frequencies of Circular Plates with Mixed Boundary Conditions

by

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Approximate natural frequencies of circular plates with mixed boundary conditions

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Abstract

The lower frequencies of a circular plate with mixed boundary conditions have been determined. For a plate, which boundary is partly clamped and partly simply supported the course of the axisymmetric lower natural frequencies have been investigated. In addition the natural frequencies of plates in asymmetric motion are presented and exhibit for each mode $m \neq 0$ two oscillation branches with quite different angular modal lines.

Keywords: circular plate, natural frequency, mixed boundary conditions, clamped, simply supported

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Zusammenfassung

Es werden die unteren Eigenfrequenzen einer Kreisplatte mit gemischten Randbedingungen bestimmt. Für eine Kreisplatte, deren Rand teilweise eingespannt und gelenkig gelagert ist, wurde der Verlauf der unteren axialsymmetrischen Eigenfrequenzen bestimmt. Asymmetrische Schwingungen zeigen jedoch zwei verschiedene Frequenzzweige mit jeweils verschiedenen angularen Knotenlinien.

1 Introduction

The problem of bending of a circular plate with mixed boundaries has been treated previously for some special conditions. A more general problem for a plate partly clamped and simply supported was formulated in [1], where forced vibrations due to harmonic load perpendicular to the plate have been treated. In addition the plate was subjected to a compressive load. In later papers [2-4] a variational approach was applied to a circular plate partially clamped and partially simply supported. One method is based upon two perturbations, i.e. one when the plate is clamped all around, the other when the plate is simply supported. The first perturbation yielded upper bounds for the eigenvalue, while the latter presented the lower bounds. There are, however, still some discrepancies of results, all of which have been restricted to the mixed boundary conditions of a partly clamped and partly simply supported system, for which only the fundamental axisymmetric mode has been determined.

With the advent of very efficient high speed computers, exhibiting the advantage of solving a large number of algebraic equations at a relatively short time, we have proposed another method for the determination of the eigenvalues of various mixed boundary conditions, where not only the fundamental natural frequency has been determined, but also higher axi- and asymmetric modes of the plate were investigated. In addition we have determined the nodal lines of the natural frequencies for the asymmetric modes.

It should be mentioned that the method is not only restricted to the determination of eigenfrequencies, but may also be applied to radially loaded plates, the response to harmonically perpendicular loads on the plates, as well as the buckling of them. In addition,

the method may also be used for other boundary combinations, as well as for rectangular plates with mixed boundary conditions.

2 Basic equations

The problem of the determination of the approximate lower natural frequencies of a circular plate exhibiting partially mixed boundary conditions along its periphery may be performed with a semi-analytical method as shown below. This method satisfying the boundary conditions for a finite number of points at the periphery $r = a$ may be applied to a large variety of boundary conditions. We shall treat here only the mixed boundary conditions of clamped, simply supported, while those of free, guided, clamped, simply supported and elastically supported edges of various peripheral edge ranges shall be treated at a later point.

The basic equations require the solution of the equation of the circular plate

$$D\Delta\Delta w + \varrho h \frac{\partial^2 w}{\partial t^2} = 0, \tag{1}$$

where the operator is

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}.$$

Equation (1) has to be solved with the appropriate mixed boundary conditions.

The bending and twisting moments are given by

$$M_r = -D\left[\frac{\partial^2 w}{\partial r^2} + \nu\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \varphi^2}\right)\right] \tag{2}$$

$$M_\varphi = -D\left[\frac{1}{r}\frac{\partial w}{\partial r} + \nu\frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \varphi^2}\right] \tag{3}$$

$$M_{r\varphi} = -D(1-\nu)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \varphi}\right) \tag{4}$$

and the transverse shearing forces are

$$Q_r = -D\frac{\partial}{\partial r}(\Delta w) \tag{5}$$

$$Q_\varphi = -D\frac{\partial}{r\partial \varphi}(\Delta w), \tag{6}$$

while the Kelvin-Kirchhoff edge reactions are given by

$$V_r = Q_r + \frac{1}{r}\frac{\partial M_{r\varphi}}{\partial \varphi} \tag{7}$$

$$V_\varphi = Q_\varphi + \frac{\partial M_{r\varphi}}{\partial r}. \tag{8}$$

The displacement of the plate is $w(r, \varphi, t)$, ϱ its density, h its thickness and $D = EIh^3/(12(1-\nu))$ its stiffness, ν is Poisson's ratio. The boundary conditions may be either

(a) clamped: $w = 0$ and $\frac{\partial w}{\partial r} = 0$ at $r = a$

(b) simply supported: $w = 0$ and $M_r = 0$ at $r = a$

(c) free: $M_r = 0$ and $V_r = 0$ at $r = a$

(d) guided: $\frac{\partial w}{\partial r} = 0$ and $V_r = 0$ at $r = a$

(e) elastically supported: $M_r - K\frac{\partial w}{\partial r} = 0$ and $V_r + kw = 0$ at $r = a$,

where K is the distributed stiffness,i.e. moment/unit length, opposing with spiral springs the edge rotation, and where k is the distributed stiffness, i.e. force/unit length, opposing the translational motion in direction w.

The solution of equation (1) yields with $w(r,\varphi,t) = W(r,\varphi)e^{i\omega t}$ the expression

$$W(r,\varphi) = \sum_{m=0}^{\infty} \left\{ A_m \mathrm{J}_m \left(\lambda \frac{r}{a}\right) + C_m \mathrm{I}_m \left(\lambda \frac{r}{a}\right) \right\} \left\{ \begin{array}{c} \cos m\varphi \\ \\ \sin m\varphi \end{array} \right\}, \tag{9}$$

where $\lambda^2 = \omega a^2 \sqrt{\varrho h/D}$. The problem of the determination of the approximate lower natural frequencies for mixed peripheral edge conditions depends on the range of the various boundary conditions at hand. It may be noticed that certain combinations of mixed boundary conditions show for the total range one and the same boundary condition, therefore reducing the effort of the numerical procedure. This may be noticed to be obvious for a combination of the following boundary conditions.

The two mixed boundary conditions clamped - simply supported exhibit at $r = a$ the same boundary condition $w = 0$ at $r = a$. For a clamped - guided mixed boundary case the total range $0 \le \varphi < 2\pi$ shows $\partial w/\partial r = 0$ at $r = a$. For a system of simply supported and free mixed boundary case the total range $0 \le \varphi < 2\pi$ has to satisfy the vanishing bending moment $M_r = 0$ at $r = a$, while for a mixed boundary condition free - guided the Kelvin-Kirchhoff edge reaction $V_r = 0$ at $r = a$, i.e.

$$V_r = -D \left[\frac{\partial}{\partial r}(\Delta w) + \frac{(1-\nu)}{r} \frac{\partial^2}{\partial r \partial \varphi} \left(\frac{1}{r} \frac{\partial w}{\partial \varphi} \right) \right] = 0, \quad \text{or}$$

$$V_r = -D \left[\frac{\partial^3 w}{\partial r^3} + \frac{1}{r}\frac{\partial^2 w}{\partial r^2} - \frac{1}{r^2}\frac{\partial w}{\partial r} + \frac{(2-\nu)}{r^2} \frac{\partial^3 w}{\partial r \partial \varphi^2} - \frac{(3-\nu)}{r^3}\frac{\partial^2 w}{\partial \varphi^2} \right] = 0. \tag{10}$$

for the total range $0 \le \varphi < 2\pi$. It may be mentioned that for a "pure" boundary condition, i.e. the vanishing range $\alpha = 0$, the well-known values λ^2 of that particular boundary condition case are obtained, while for the "pure" boundary condition of the other range, i.e. that at $\alpha = 2\pi$, the values λ^2 of that particular boundary condition case are obtained. For mixed boundary condition cases of varying $\alpha \neq 0$ and $\alpha \neq 2\pi$ the

approximate λ^2-values are obtained from the following procedure and require for different modes m and n careful selection of the number of points, at which the remaining boundary conditions have to be satisfied to yield acceptable final results. More shall be remarked in the numerical solution below.

3 Method of solution

Let us treat a plate clamped along part of the boundary and simply supported along the remainder, requiring only two different boundary conditions for the numerical procedure (Fig. 1). If the plate is clamped in the range $0 < \varphi < \alpha$ and simply supported in the range $\alpha \leq \varphi \leq 2\pi$, the plate exhibits for the total range $0 \leq \varphi < 2\pi$ a vanishing deflection $w = 0$ at $r = a$ and exhibits therefore the solution

$$W(r,\varphi) = \sum_{m=0}^{\infty} \left[\mathrm{J}_m\left(\lambda\frac{r}{a}\right) - \frac{\mathrm{J}_m(\lambda)}{\mathrm{I}_m(\lambda)}\mathrm{I}_m\left(\lambda\frac{r}{a}\right)\right]\{A_m \cos m\varphi + B_m \sin m\varphi\}\,. \tag{11}$$

This satisfies $W(a,\varphi) = 0$ in the total range $0 \leq \varphi < 2\pi$. If $\alpha = 2\pi$, i.e. the plate is totally clamped the second boundary condition $\partial w/\partial r = 0$ at $r = a$ yields

$$\mathrm{J}'_m(\lambda) - \frac{\mathrm{J}_m(\lambda)}{\mathrm{I}_m(\lambda)}\mathrm{I}'_m(\lambda) = 0 \tag{12}$$

and we obtain for the determination of λ_{mn} the vanishing determinant

$$\begin{vmatrix} \mathrm{J}_m(\lambda) & \mathrm{I}_m(\lambda) \\ \mathrm{J}'_m(\lambda) & \mathrm{I}'_m(\lambda) \end{vmatrix} = 0 \tag{13}$$

which yields the squares of the eigenvalues $\lambda_{mn}^{(c)2}$ for a clamped plate given in table 1.

If the plate is totally simply supported, the value $\alpha = 0$ and equation (11) has to satisfy in addition the vanishing bending moment $M_r = 0$ at the edge $r = a$. This yields with

(2)

$$\left[\mathrm{J}''_m(\lambda) + \frac{\nu}{\lambda}\mathrm{J}'_m(\lambda)\right]\mathrm{I}_m(\lambda) - \left[\mathrm{I}''_m(\lambda) + \frac{\nu}{\lambda}\mathrm{I}'_m(\lambda)\right]\mathrm{J}_m(\lambda) = 0 \tag{14}$$

and the determination of λ requires the solution of the determinant

$$\begin{vmatrix} \mathrm{J}_m(\lambda) & \mathrm{I}_m(\lambda) \\ \mathrm{J}''_m(\lambda) + \frac{\nu}{\lambda}\mathrm{J}'_m(\lambda) & \mathrm{I}''_m(\lambda) + \frac{\nu}{\lambda}\mathrm{I}'_m(\lambda) \end{vmatrix} = 0, \tag{15}$$

which yields for $\nu = 0.3$ the squares of the eigenvalues $\lambda_{mn}^{(ss)2}$ for a simply supported plate given in table 2.

For a plate of partly clamped boundary in the range $0 < \varphi < \alpha$ and a simply supported boundary in the range $\alpha \leq \varphi \leq 2\pi$ (Fig. 1) we have to satisfy $\partial w/\partial r = 0$ at $r = a$, i.e.

$$\sum_{m=0}^{\infty}\left[\mathrm{J}'_m(\lambda) - \frac{\mathrm{J}_m(\lambda)}{\mathrm{I}_m(\lambda)}\mathrm{I}'_m(\lambda)\right](A_m\cos m\varphi + B_m \sin m\varphi) = 0 \quad \text{in the range } 0 < \varphi < \alpha \tag{16}$$

and $M_r = 0$, i.e.

$$\sum_{m=0}^{\infty}\left\{\left[\mathrm{J}''_m(\lambda) + \frac{\nu}{\lambda}\mathrm{J}'_m(\lambda)\right] - \frac{\mathrm{J}_m(\lambda)}{\mathrm{I}_m(\lambda)}\left[\mathrm{I}''_m(\lambda) + \frac{\nu}{\lambda}\mathrm{I}'_m(\lambda)\right]\right\}(A_m\cos m\varphi + B_m\sin m\varphi) = 0$$

$$\text{in the range } \alpha \leq \varphi \leq 2\pi. \tag{17}$$

The equations (16) and (17) have to be satisfied at a chosen number of points in each range. If $\varphi = \alpha n_1/(N_1+1)$ with $n_1 = 1, 2, \ldots, N_1$ in the range $0 < \varphi < \alpha$ and $\varphi = \alpha + (2\pi - \alpha)n_2/N_2$ with $n_2 = 0, 1, \ldots, N_2$ in the range $\alpha \leq \varphi \leq 2\pi$, the equations (16) and (17) read then ($N_1 + N_2$ even)

$$\sum_{m=0}^{(N_1+N_2)/2}\left[\mathrm{J}'_m(\lambda) - \frac{\mathrm{J}_m(\lambda)}{\mathrm{I}_m(\lambda)}\mathrm{I}'_m(\lambda)\right]\left\{A_m\cos\left(\frac{m\alpha n_1}{N_1+1}\right) + B_m\sin\left(\frac{m\alpha n_1}{N_1+1}\right)\right\} = 0$$

$$\text{for } n_1 = 1, 2, \ldots, N_1 \tag{18}$$

and

$$\sum_{m=0}^{(N_1+N_2)/2} \left\{ \left[\mathrm{J}''_m(\lambda) + \frac{\nu}{\lambda} \mathrm{J}'_m(\lambda) \right] - \frac{\mathrm{J}_m(\lambda)}{\mathrm{I}_m(\lambda)} \left[\mathrm{I}''_m(\lambda) + \frac{\nu}{\lambda} \mathrm{I}'_m(\lambda) \right] \right\} \times$$
$$\times \left\{ A_m \cos\left(m \left[\alpha + \frac{(2\pi - \alpha) n_2}{N_2} \right] \right) + B_m \sin\left(m \left[\alpha + \frac{(2\pi - \alpha) n_2}{N_2} \right] \right) \right\} = 0$$

for $n_2 = 0, 1, \ldots, N_2$. (19)

The equation (18) represents N_1 homogeneous algebraic equations in the unknowns $A_0, A_1, \ldots, A_{(N_1+N_2)/2}$ and $B_1, B_2, \ldots, B_{(N_1+N_2)/2}$, while equation (19) yields $N_2 + 1$ equations in those constants. The vanishing coefficient determinant represents the equation for the determination of the lower approximate eigenvalues λ. It yields the lower values λ_{mn} for a given magnitude of α in ascending order of which for a given mode m the values λ^2 shall be located between those of the totally clamped case and those of the totally simply supported case.

4 Numerical evaluations and conclusions

The above procedure has been numerically evaluated for the lower modes of a circular plate, i.e. for $m = 0$, 1 and 2 and $n = 1, 2$. Figure 2 represents $\lambda_{mn}^2 = \omega a^2 \sqrt{\varrho h / D}$ as a function of the angle α/π. For $\alpha = 0$ we exhibit the plate with a purely simply supported boundary, while for $\alpha = 2\pi$ the boundary is in a purely clamped state. These values are indicated in the graph as $\circ$ for the simply supported boundary and as $\otimes$ for the clamped boundary. The influence of the Poisson ratio ν, which appears only in the simply supported boundary is indicated by the $*$-sign where the upper star represents the λ-value for $\nu = 0.5$ and the lower one that of $\nu = 0.2$. The square of the eigenvalue

λ_{01}^2 increases from 4.9351 to 10.2158 and exhibits a slight flexion close to $\alpha = \pi$. The here presented numerical results were compared with those of reference [3] and [4], where only the mode $m = 0$, $n = 1$ has been treated and presented. The value presented in Fig. 2 for $m = 0$ and $n = 1$ show a very close proximity to that given in [3]. The numerical evaluation in our treatment required for $N_1 + N_2 = 150$, in which N_1 and N_2 vary according to the magnitude of α. The magnitudes of N_1 and N_2 are chosen such that an increase of these magnitudes do no longer affect the accuracy of the plotted λ^2-values. If α is small the portion of the boundary being clamped needs only a small value for N_1, while that of the boundary being simply supported requires a large number N_2 for the numerical procedure. As the clamped portion α of the boundary increases, so does the number N_1, while N_2 decreases, such that the sum $N_1 + N_2 = 150$, as indicated in Fig. 2. Treating the axisymmetric mode $m = 0$ further yields the results shown as the curve $m = 0$, $n = 2$ (second axisymmetric mode), starting for a totally simply supported plate at $\lambda_{02}^{2(ss)} = 29.72$ and reaches for totally clamped plate the value $\lambda_{02}^{2(c)} = 39.77$ (see also Table 1 and 2). In the case of the eigenvalue λ_{02}^2 we notice increased flexion as α increases. The investigation of the asymmetric mode $m = 1$ and $m = 2$ exhibits for the first modal number $n = 1$ two values λ_{12} and λ_{21} in the range $0 < \alpha < 2\pi$. For $m = 1$ and $n = 1$ the value of λ_{11}^2 yields for a totally supported plate the magnitude $\lambda_{11}^{2(ss)} = 13.8982$ and for a completely clamped plate the value $\lambda_{11}^{2(c)} = 21.26$. In the mixed boundary range $\alpha \neq 0, 2\pi$ we have to distinguish two modes, which have been indicated by 1 and 2. The mode 1 exhibits in the range $0 < \alpha < \pi$ eigenvalues λ_{11}^2 being smaller than those of the branch 2, while above $\alpha > \pi$, i.e. the case, where more than half of the plate is clamped at its boundary, the eigenvalue λ_{11}^2 is larger than that of the branch 2. Those

two branches caused by the mixing of boundaries shall not only exhibit different natural frequencies, but also different nodal lines, as we shall see later. Treating now the mode $m = 2$, $n = 1$, which shall start for a completely simply supported plate with the square of the eigenvalue, i.e. $\lambda_{21}^{2(ss)} = 25.61$ and reaches for the totally clamped plate the magnitude $\lambda_{21}^{2(c)} = 34.877$, exhibits, also two branches ① and ② with increased fluctuations. In Fig. 3 we present the fundamental axisymmetric mode $m = 0$, $n = 1$ again, where $N_1 + N_2 = 160$ and ν was chosen to be $\nu = 0.2$, 0.3 and 0.5. If the Poisson ratio ν is larger the natural frequency is larger, the magnitude of which decreases as α increases. A similar behavior is found for a Poisson ratio ν being smaller. Figure 4 shows the numerical results for the mode $m = 1$, $n = 2$, which exhibits again two branches for the natural frequency of the plate for $\alpha \neq 0$ and 2π. Both curves start for $\alpha = 0$ at $\lambda_{12}^{2(ss)} = 48.48$ and end for $\alpha = 2\pi$ at $\lambda_{12}^{2(c)} = 60.83$.

In Fig. 5 we represent the nodal lines for the branches ① and ② of the mode $m = n = 1$ with the angle α as parameter in degrees. Behind each α the magnitude of the dimensionless natural frequency $\lambda^2 = \omega a^2 \sqrt{\varrho h/D} = \omega a^2 \sqrt{12\varrho(1-\nu)/EIh^2}$ is indicated. The drawing of the node lines was done as follows. For chosen α-value, mode numbers m, n and branch the λ^2-value, i.e. the eigenvalue, and the coefficients A_m, B_m, i.e. the eigenvector, are computed from (18) and (19). Introducing these results into (11) and demanding $W(r, \varphi) = 0$ yields for chosen r/a with $(0 < r/a < 1)$ the appropriate angle $\varphi = \varphi(r)$. With increasing angle α, i.e. increasing portion of the clamped boundary the nodal line turns in counter-clockwise direction, as indicated. For a totally simply supported plate the nodal line is located at $\varphi = 0$. This is also valid for a completely clamped plate, while for a plate of which half of the boundary is clamped and the other

half simply supported, the nodal line is located at $\varphi = \pi/2$, i.e. perpendicular to that of the pure boundary cases. If only one quarter of the boundary is clamped, i.e. $\alpha = \pi/2$, then the nodal line is at $\varphi = \pi/4$. The nodal lines for branch ② of the mode $m = n = 1$ are also presented in Fig. 5. These nodal lines appear as soon as $\alpha \neq 0,\ 2\pi$. For a small clamped part of the boundary of $\alpha = 10°$, for which $\lambda_{11}^2 = 14.80$ the nodal line exhibits a small curvature and does no longer pass through the center of the plate. As α increases the curvature of the nodal line increases and rotates as a whole in counter-clockwise direction, exhibiting increased curvature especially at the area close to the boundary. While for the branch ① and $\alpha = \pi$ the nodal line is a straight line $\phi = \pi/2$, that of the branch ② shall be located as a curved line in the lower half (i.e. the simply supported part) of the plate and shall be below the center of the plate with a larger inward curvature close to the boundary. This curvature will increase strongly with the increase of the clamped part α.

For the branches ① and ② of the mode $m = 2$, $n = 1$ the nodal lines are presented in Fig. 6. If the clamped portion of the boundary is a small $\alpha = 10°$ the nodal lines of branch ① are the two straight and perpendicular lines ① as indicated in Fig. 6a, rotated in a counter-clockwise direction by 5°. The nodal lines associated with branch ② are the exhibited hyperbola-like curves. A further increase of α, meaning a larger portion of the clamped boundary yields a counter-clockwise rotation of the nodal curves ① and ②. It may be noticed that perpendicular nodal line of the branch ① shifts from the center of the plate and exhibits slight curvature at the boundary of the plate (see Fig. 6c for $\alpha = 120°$). In addition the nodal lines of branch ② show at those nodal curves, close to the clamped boundary increasing downward curvatures, as α increases. For larger α-values the nodal lines are presented in Figs. 6d through 6g.

List of symbols

a	radius of circular plate
D	Stiffness of plate ($D = \frac{EIh^3}{12(1-\nu)}$)
E	modulus of elasticity
h	thickness of plate
I	moment of inertia
I_m	modified Bessel function
J_m	Bessel function
k	distributed stiffness of translational springs (force/unit length)
K	distributed stiffness of spiral springs (moment/unit length)
M	bending moment, twisting moment
N_1, N_2	number of points, at which the boundary conditions are satisfied
Q	transverse shearing force
r, φ	polar coordinates
t	time
V	Kelvin-Kirchhoff edge reaction
$w(r, \varphi, t)$	displacement of plate
α	angle
ϱ	mass density of plate
ν	Poisson's ratio
λ	eigenvalue

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Table 1: Squares of the eigenvalues $\lambda_{mn}^{(c)2}$ for a clamped plate

Table 2: Squares of the eigenvalues $\lambda_{mn}^{(ss)2}$ for a simply supported plate for Poisson ratio $\nu = 0.3$

n \ m	0	1	2	3
1	10.2158	21.2604	34.8770	51.0300
2	39.7711	60.8287	84.5826	111.0214
3	89.1041	120.0792	153.8151	190.3038
4	158.1842	199.0534	242.7206	289.1799

Table 1: $\lambda_{mn}^{(c)2}$ for a clamped plate

n \ m	0	1	2	3
1	4.9351	13.8982	25.6133	39.9573
2	29.7200	48.4789	70.1170	94.5490
3	74.1561	102.7733	134.2978	168.6749
4	138.3181	176.8012	218.2026	262.4847

Table 2: $\lambda_{mn}^{(ss)2}$ for a simply supported plate ($\nu = 0.3$)

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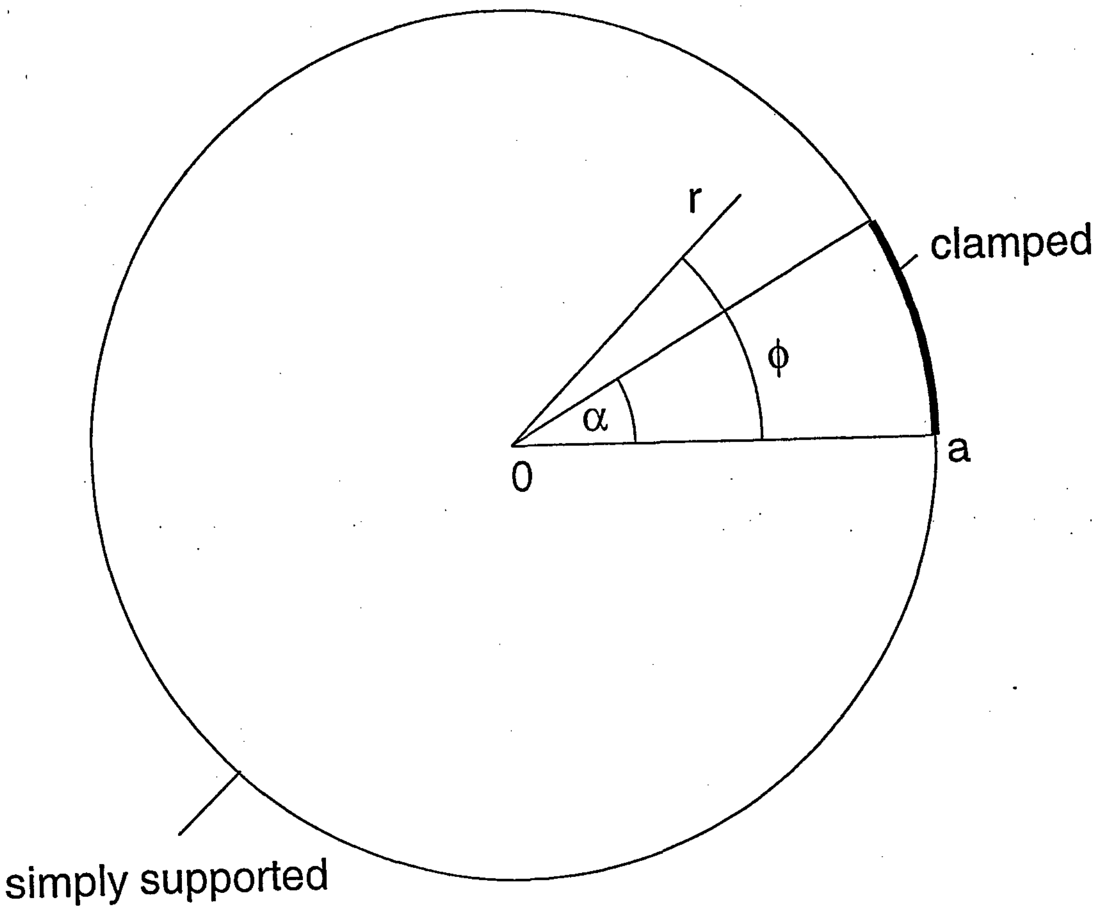

Fig. 1 H. F. Bauer/ W. Eidel

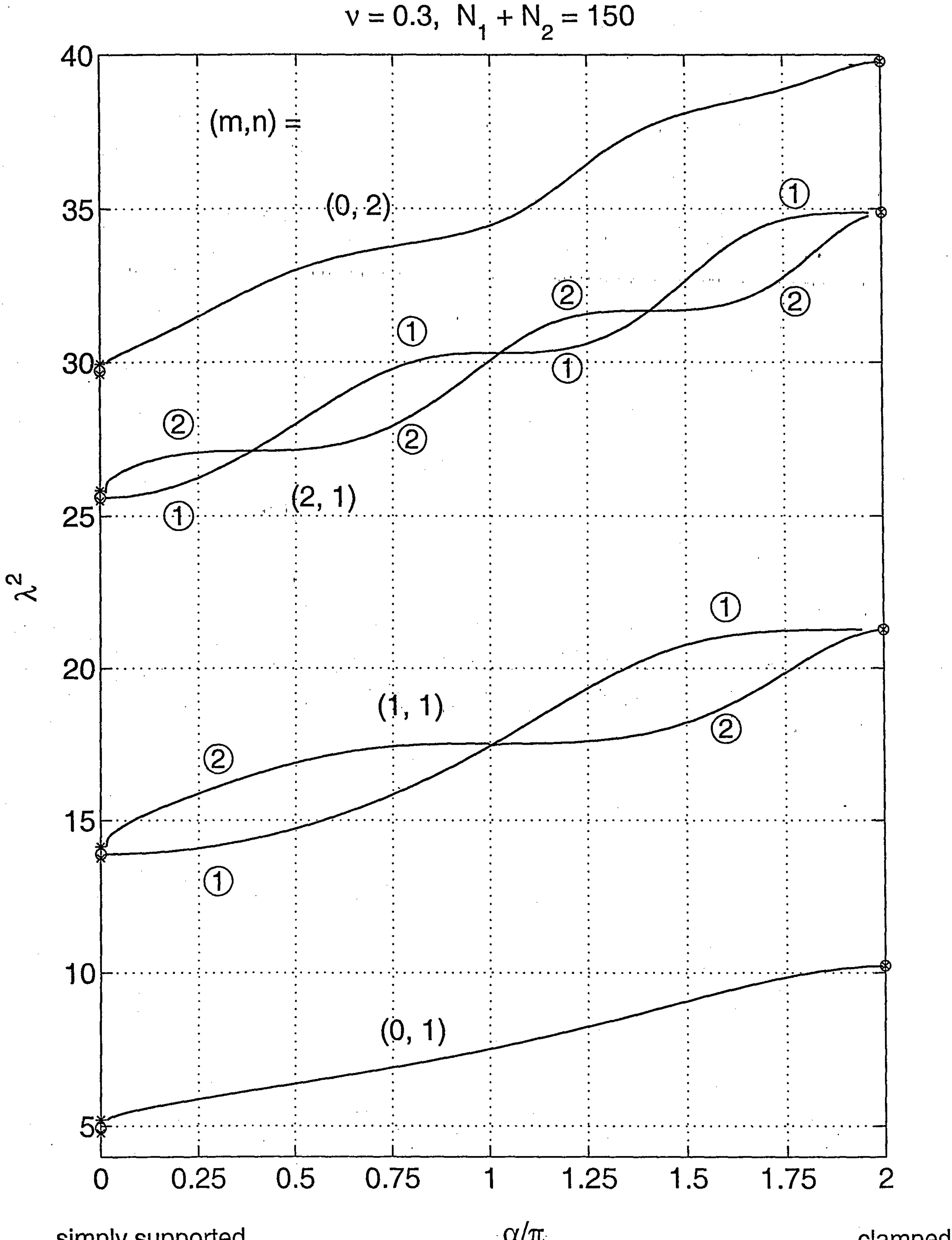

Fig. 2 H. F. Bauer/ W. Eidel

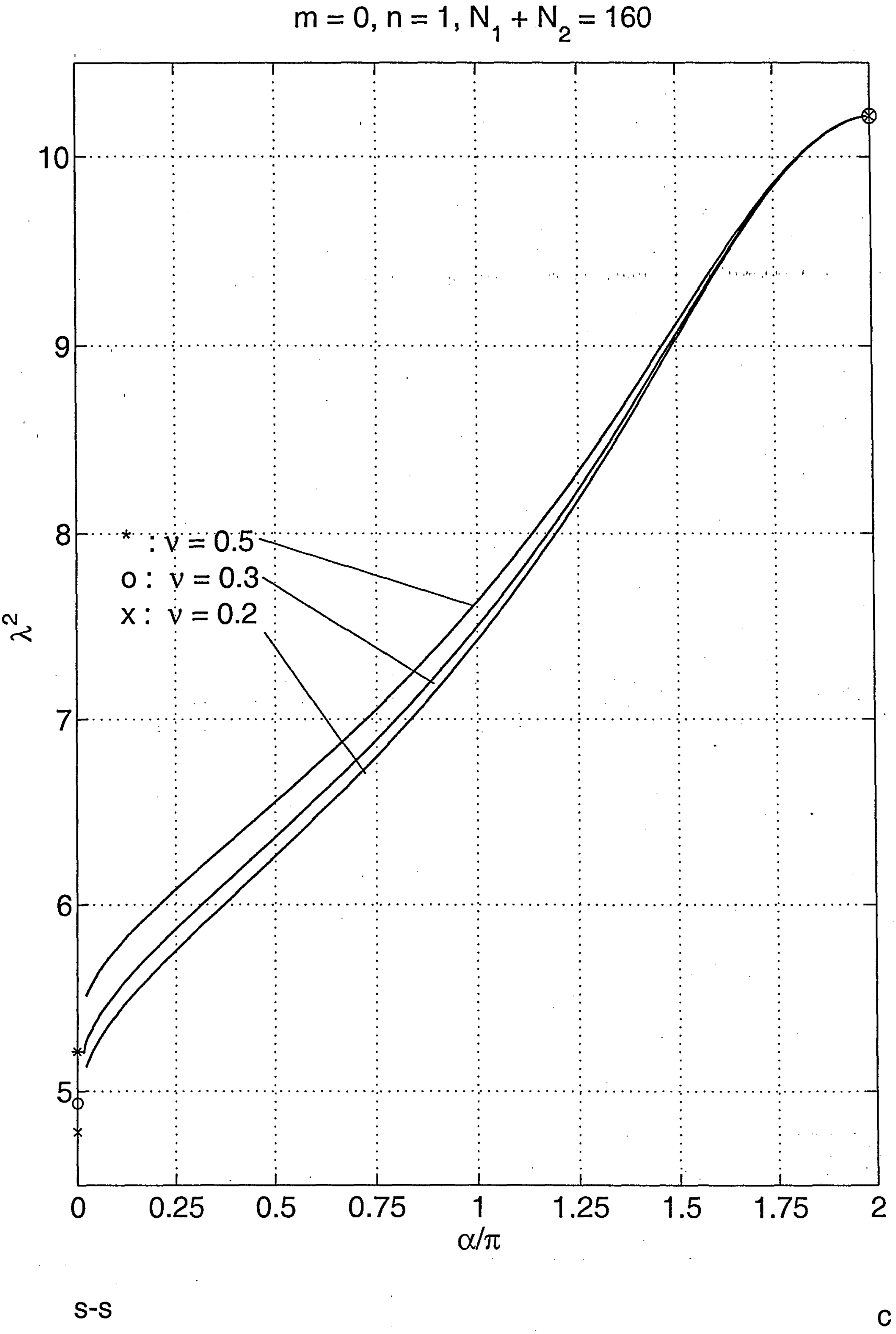

Fig. 3 H. F. Bauer/ W. Eidel

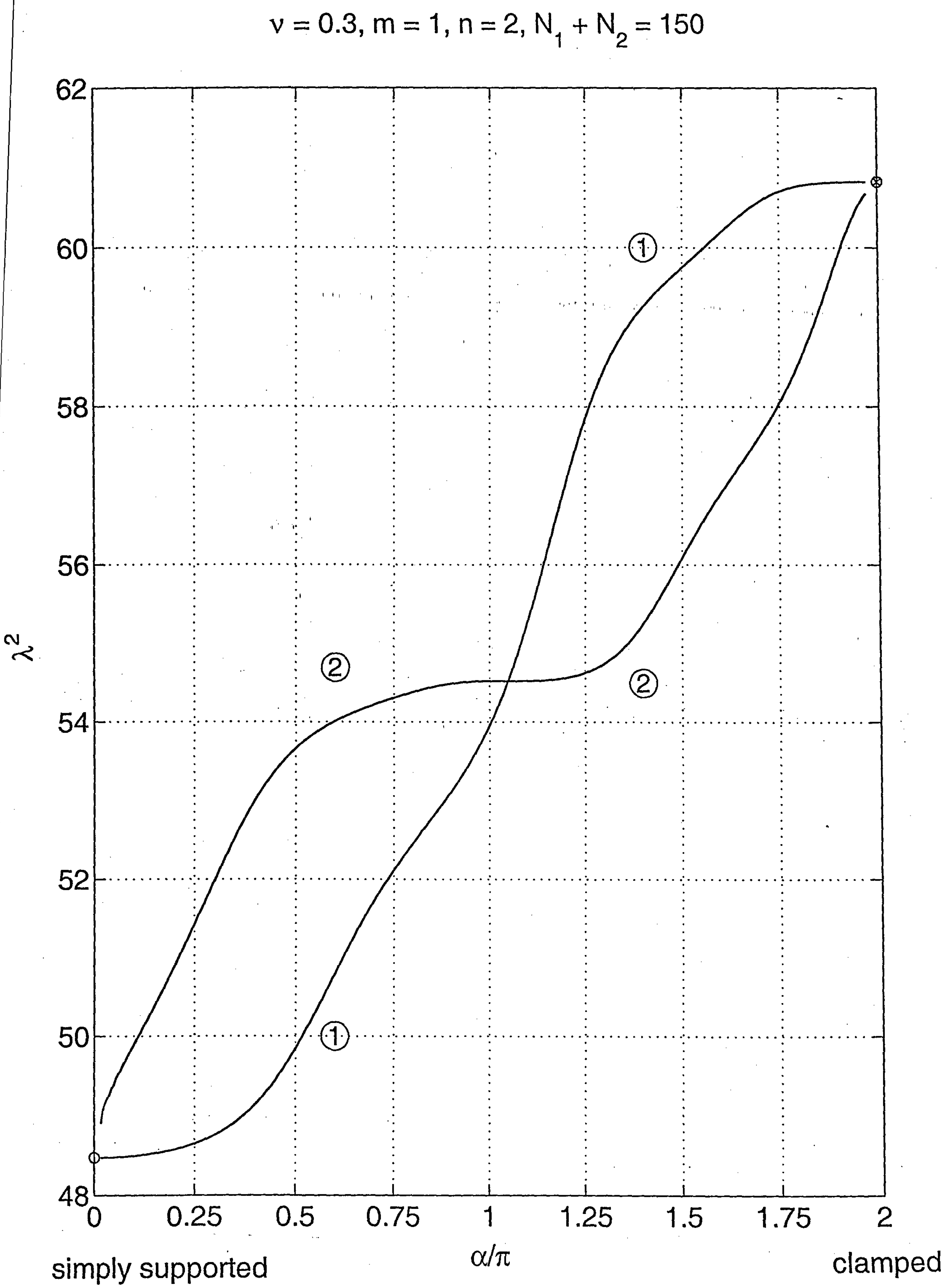

Fig. 4 H. F. Bauer/ W. Eidel

$\alpha = 10°, \nu = 0.3, m = n = 1$

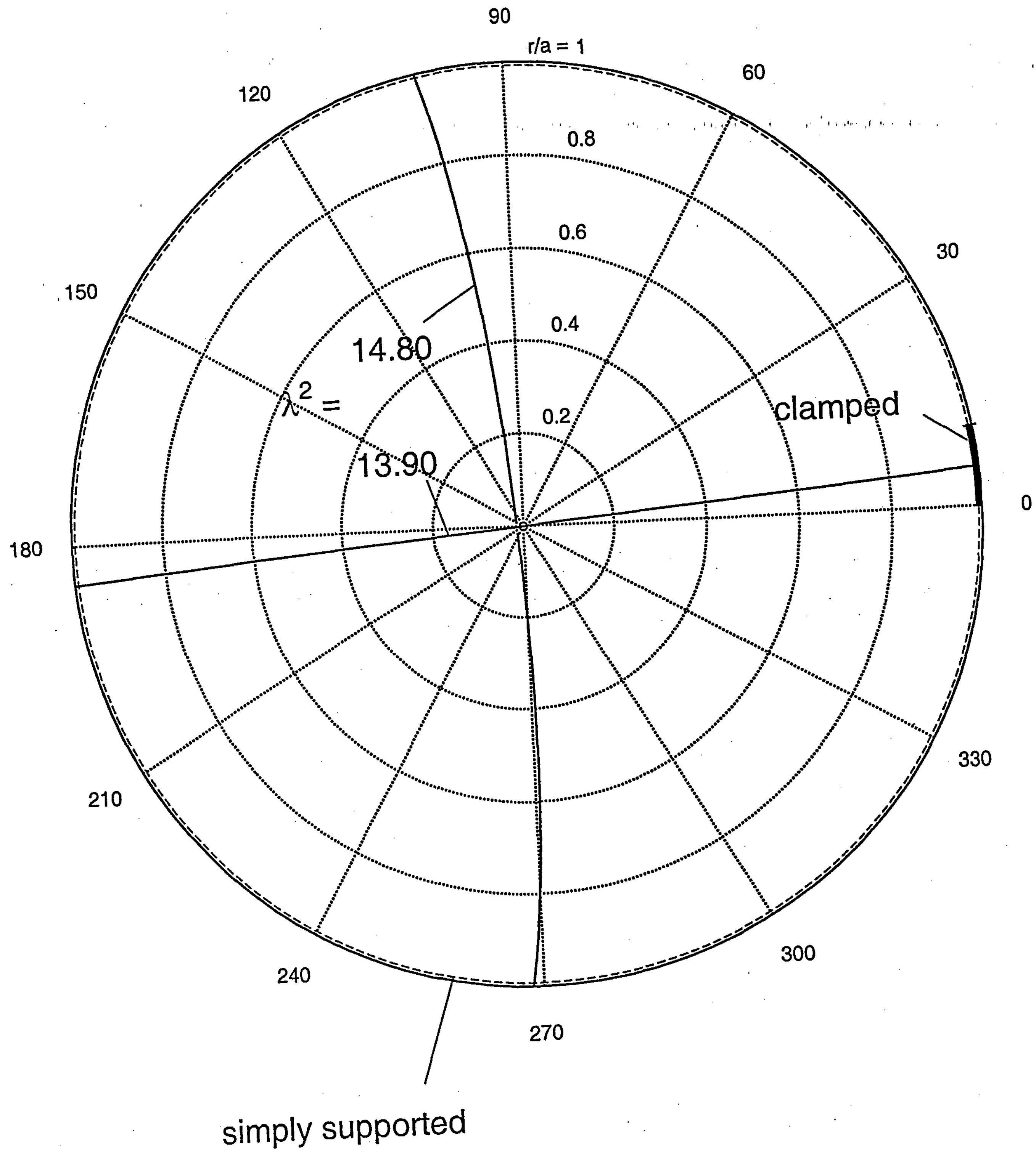

Fig. 5a H. F. Bauer/ W. Eidel

$\alpha = 20°, \nu = 0.3, m = n = 1$

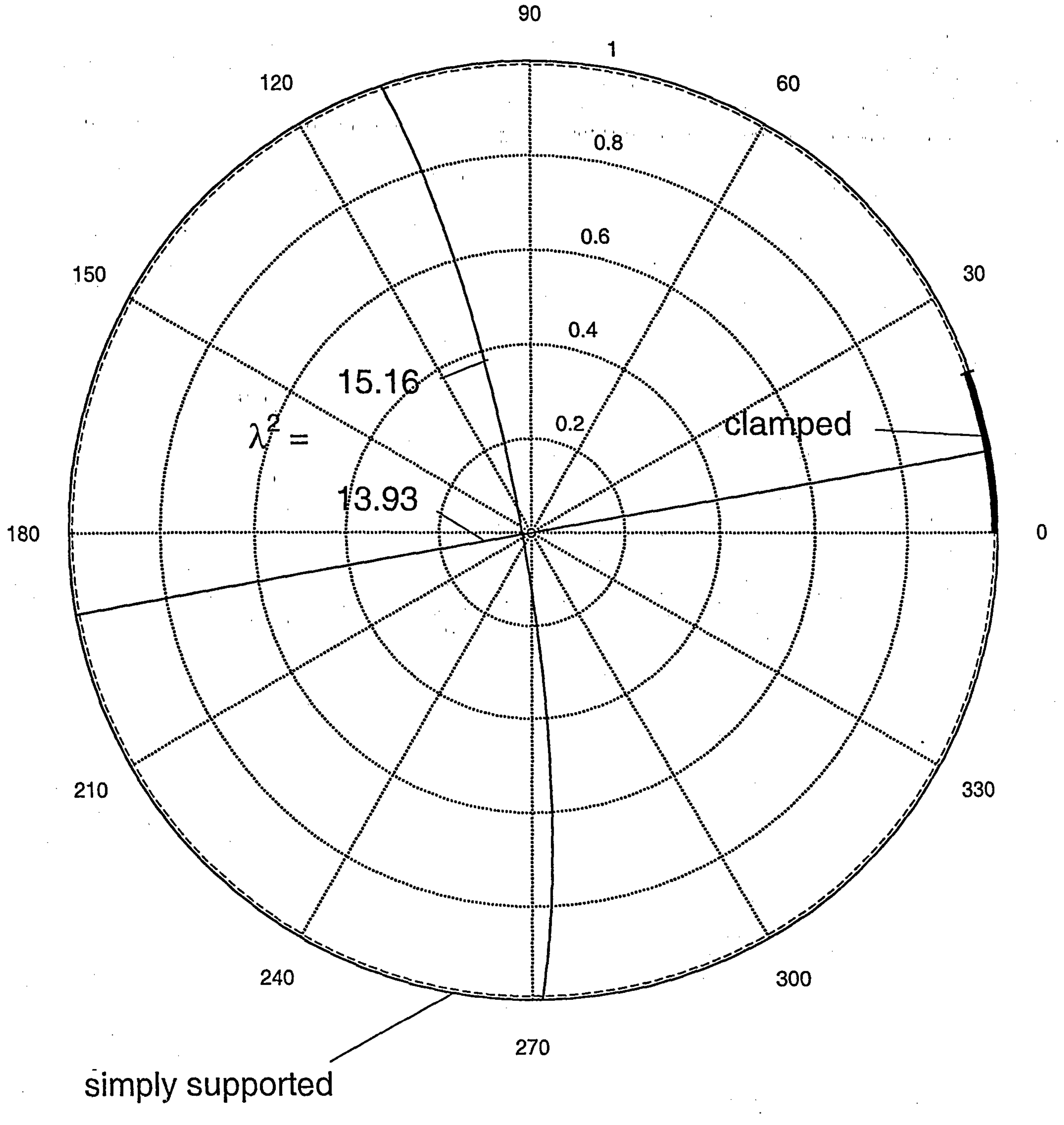

Fig. 5b H. F. Bauer/ W. Eidel

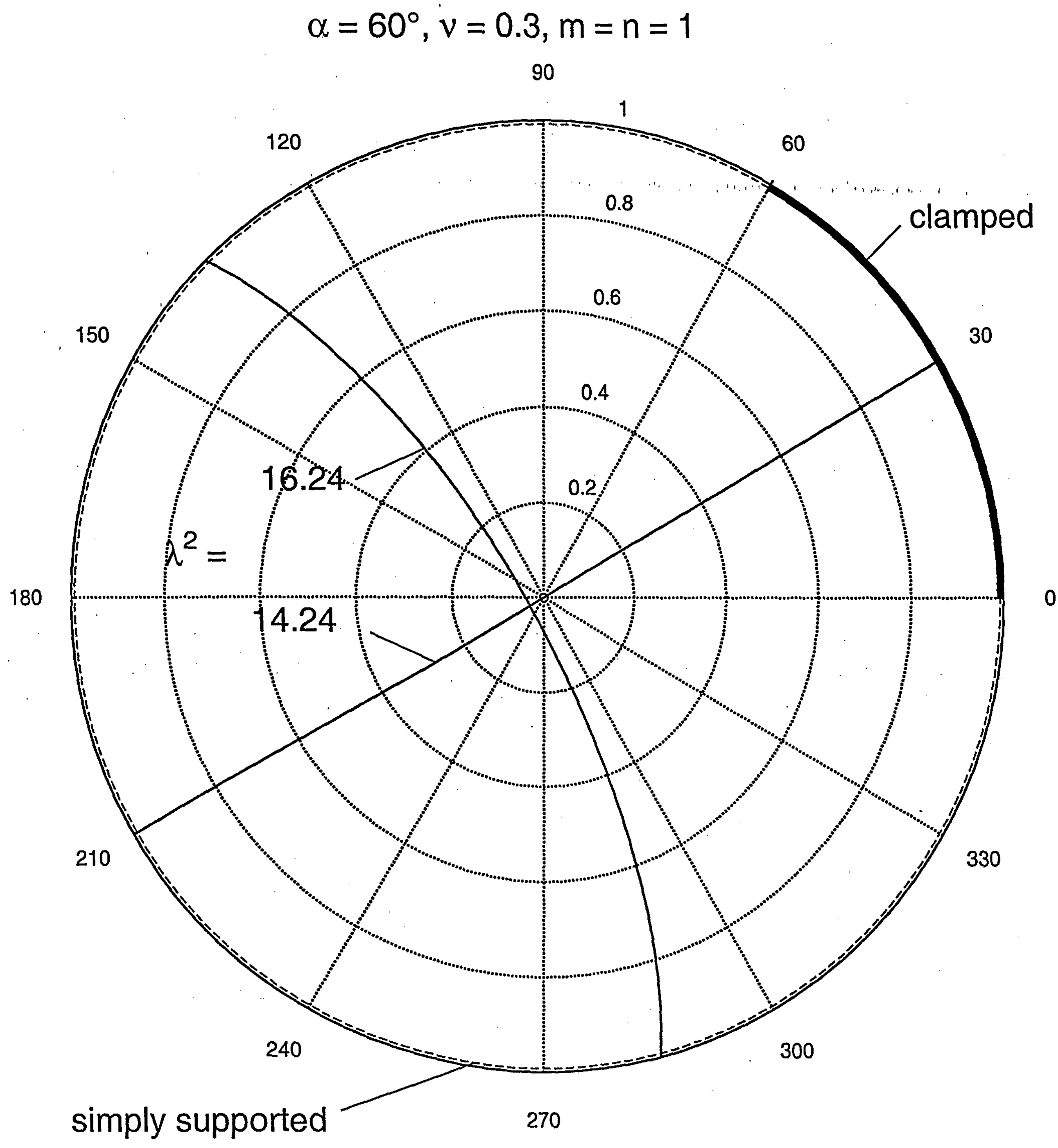

Fig. 5c H. F. Bauer/ W. Eidel

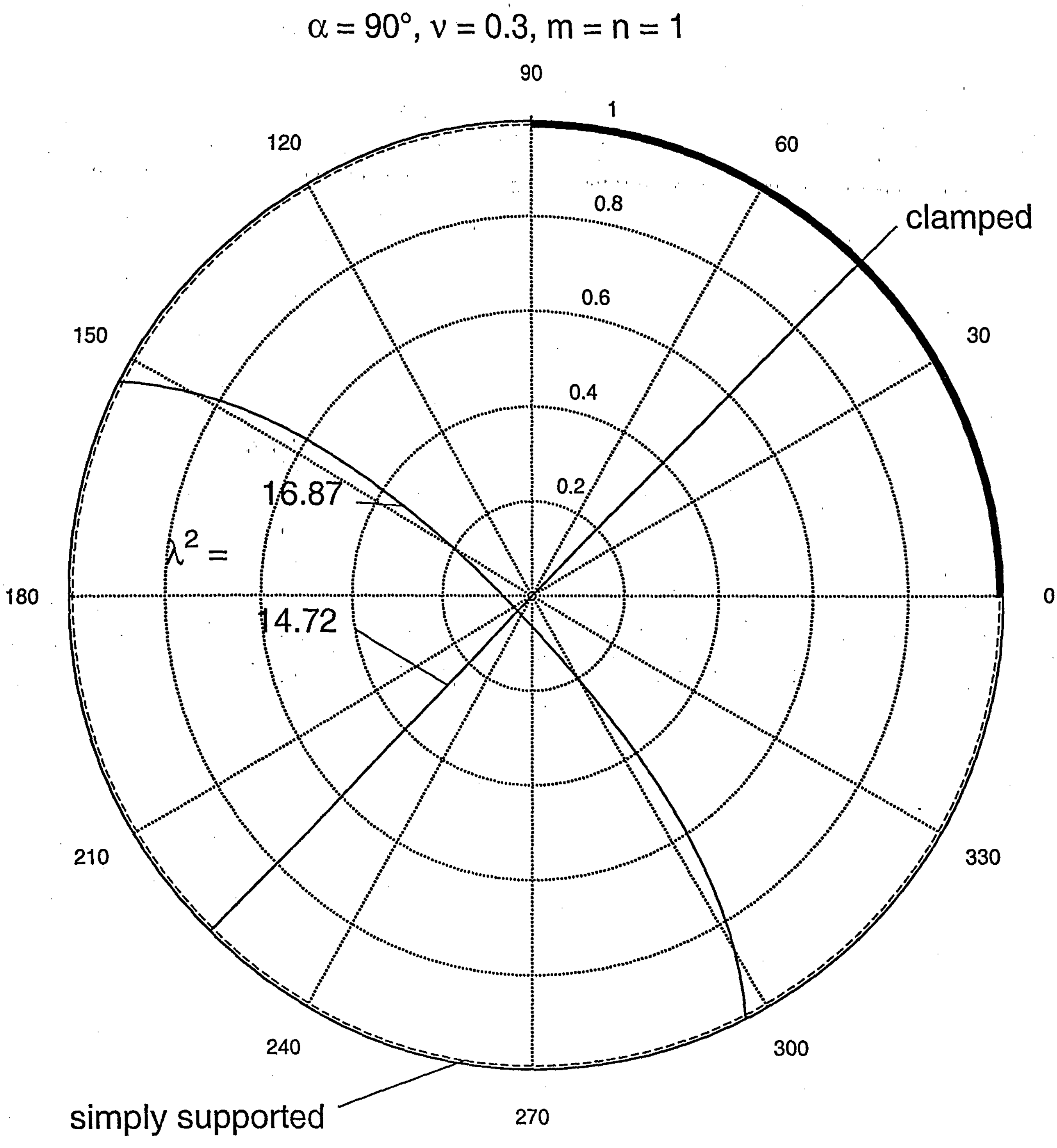

Fig. 5d H. F. Bauer/ W. Eidel

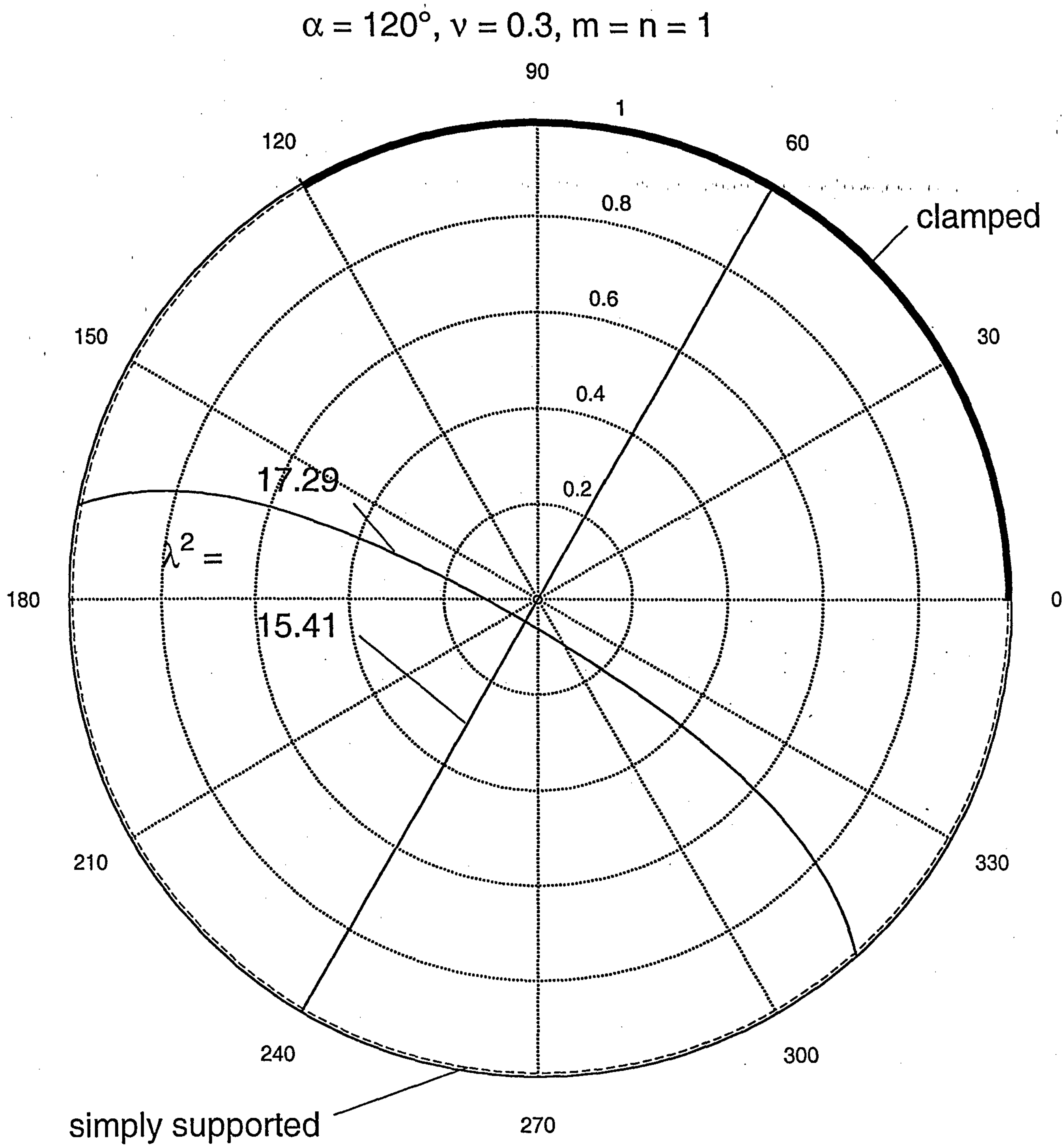

Fig. 5e H. F. Bauer/ W. Eidel

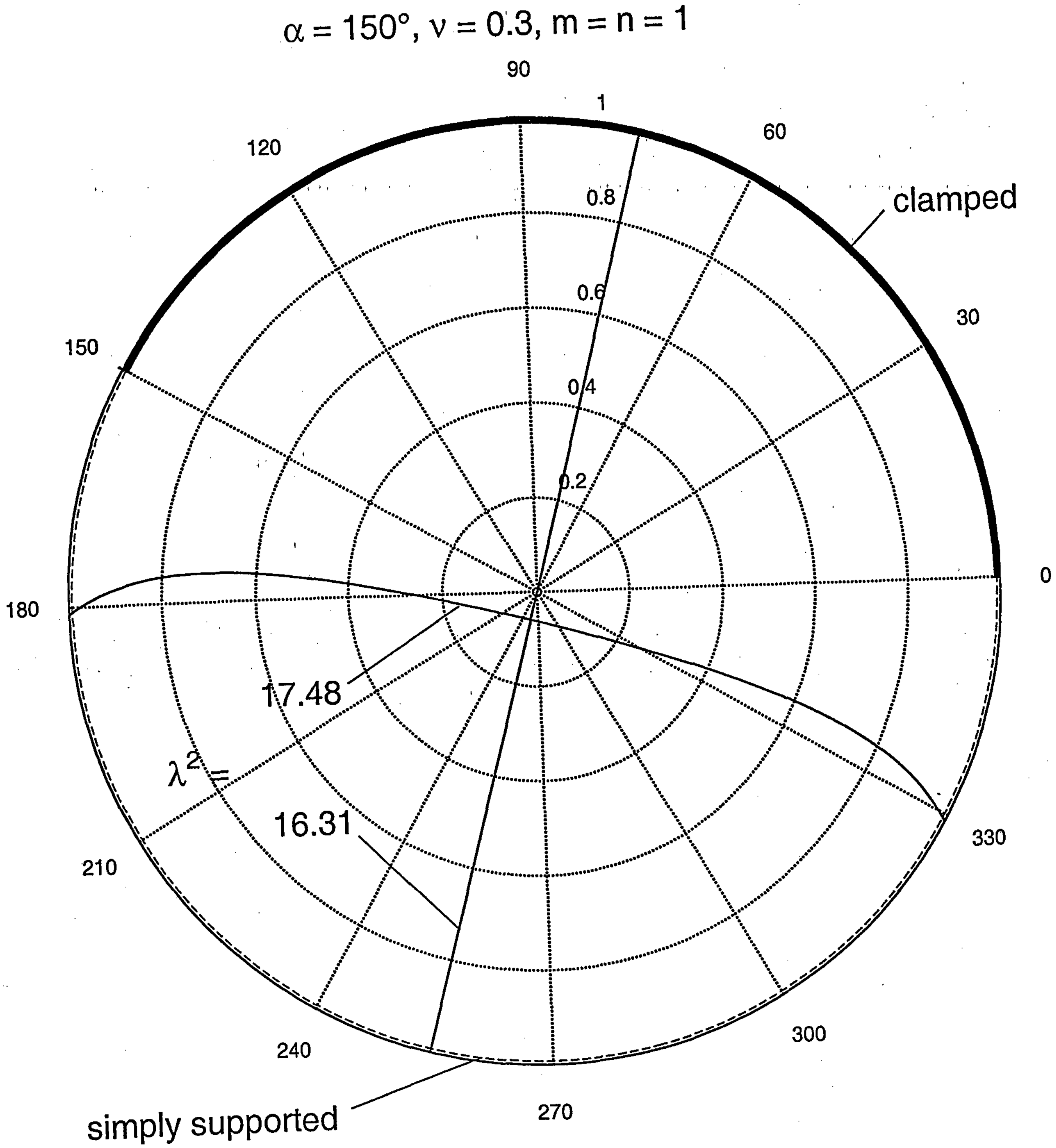

Fig. 5f H. F. Bauer/ W. Eidel

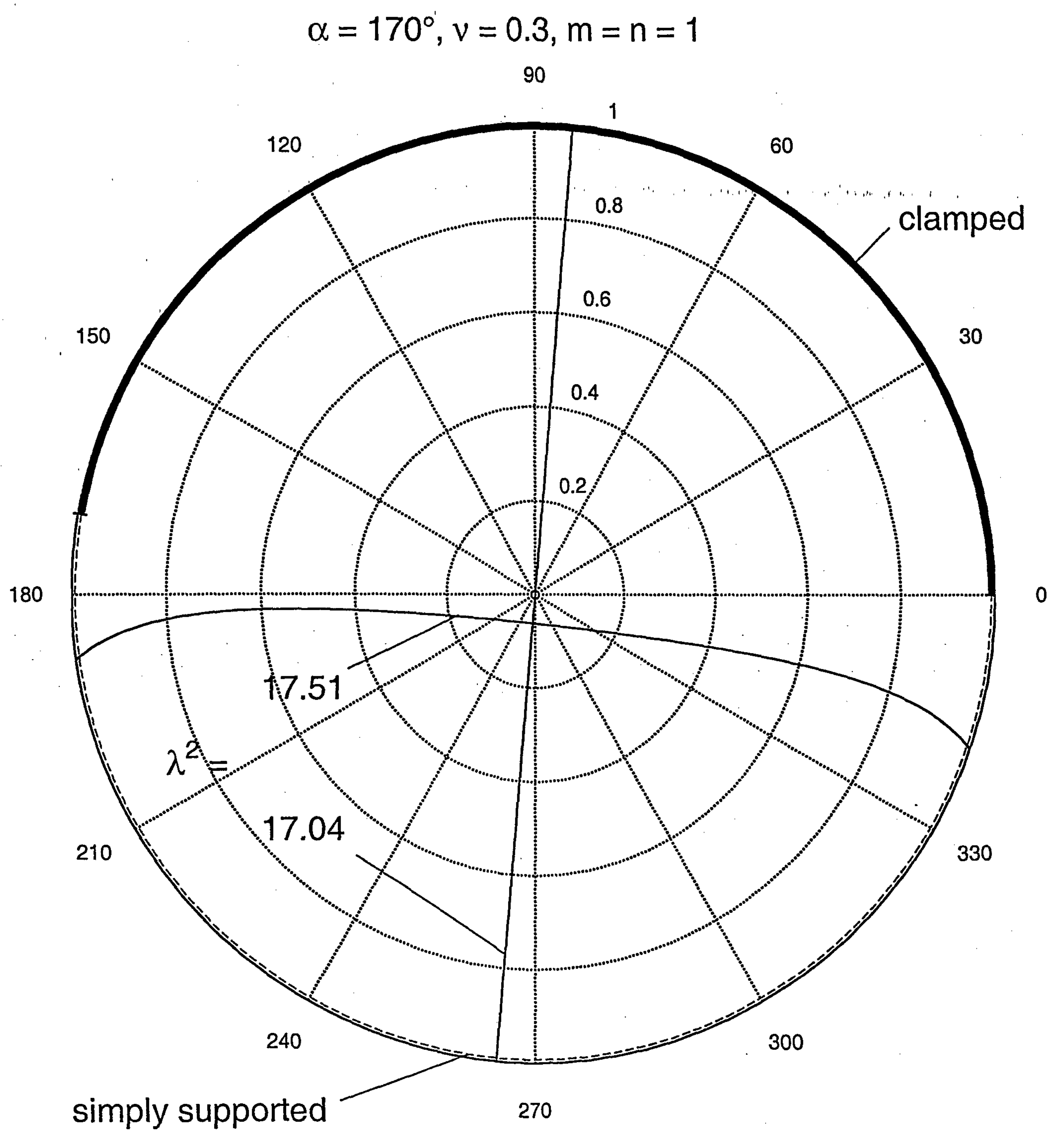

Fig. 5g H. F. Bauer/ W. Eidel

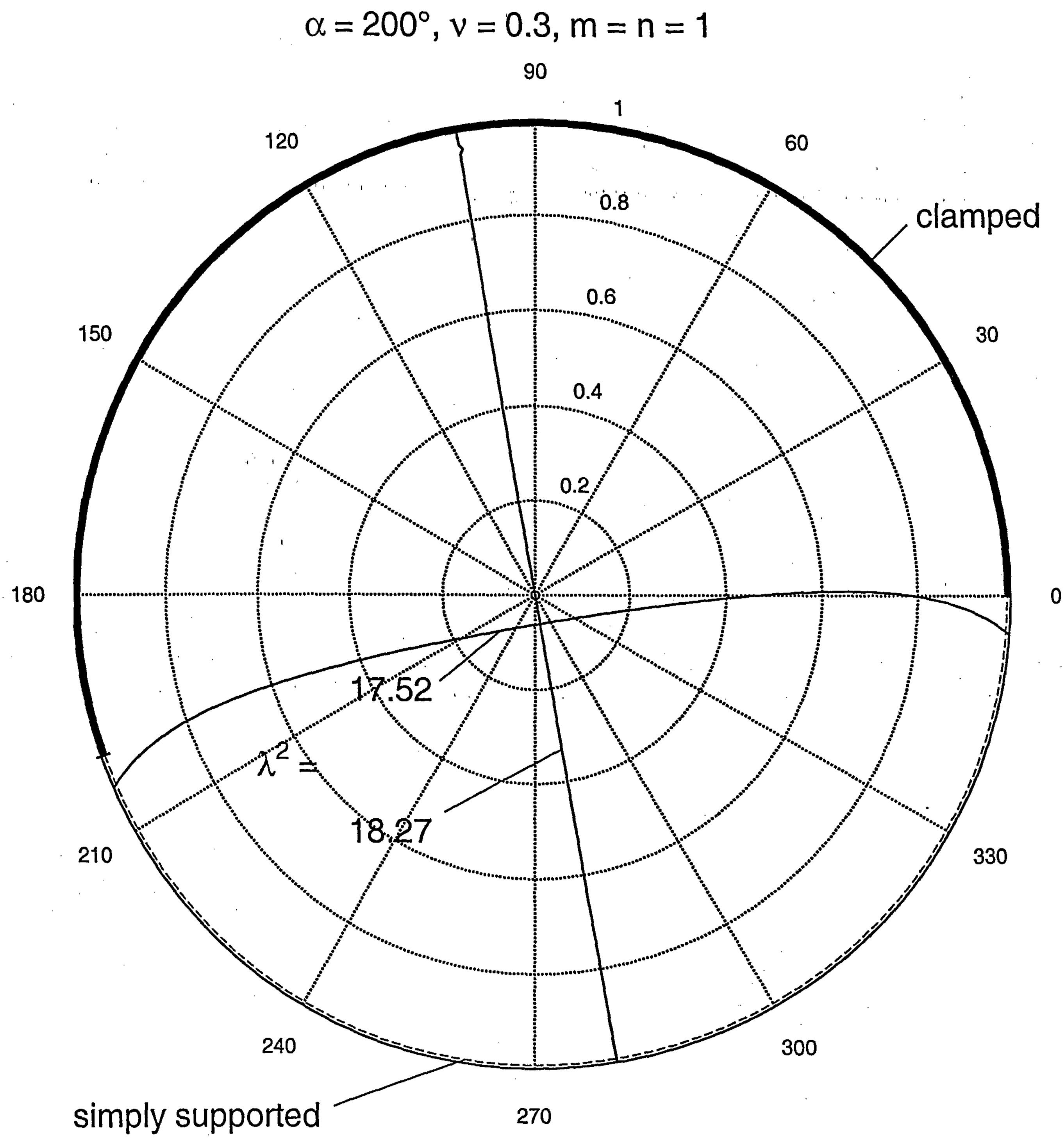

Fig. 5h H. F. Bauer/ W. Eidel

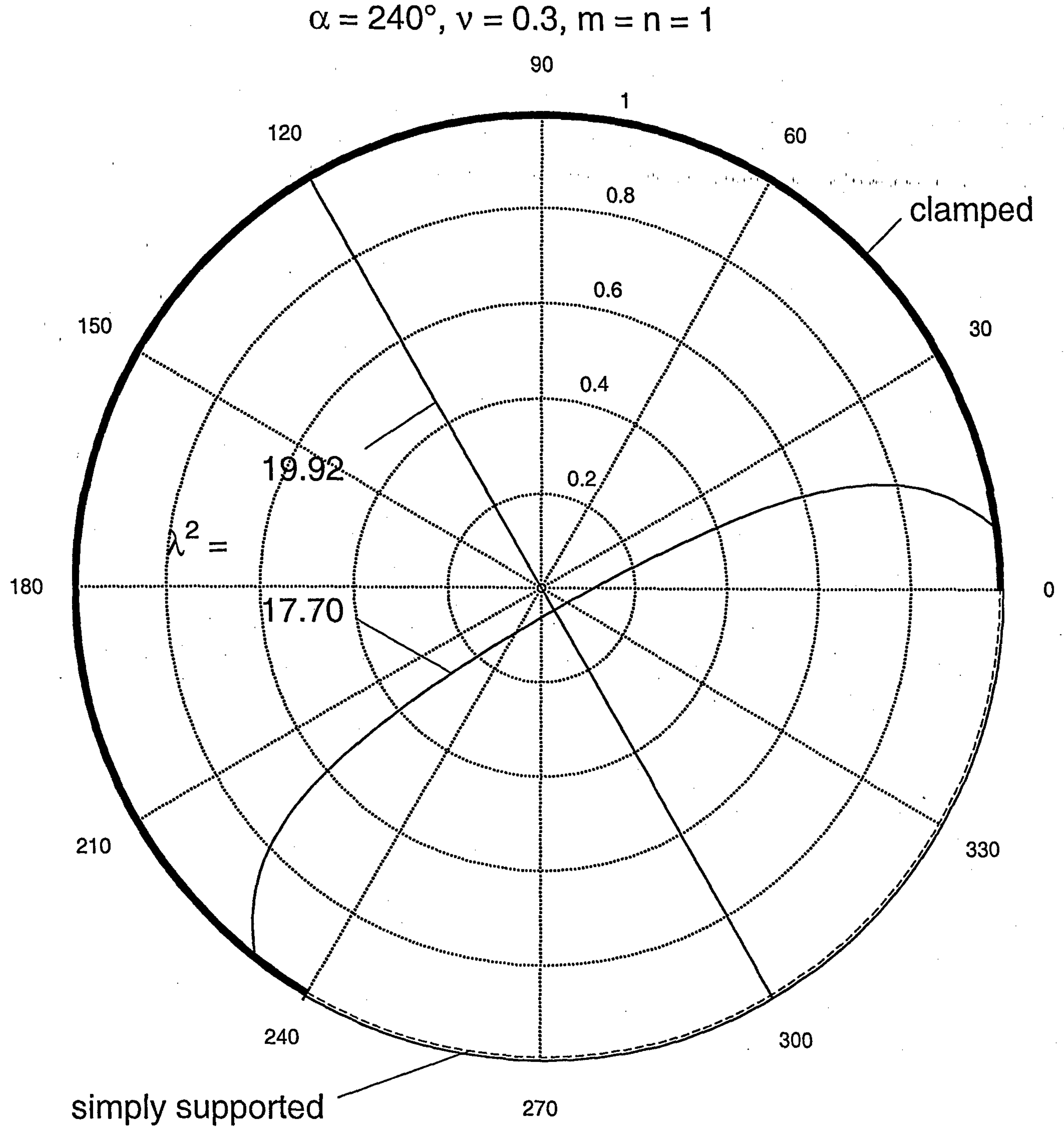

Fig. 5i H. F. Bauer/ W. Eidel

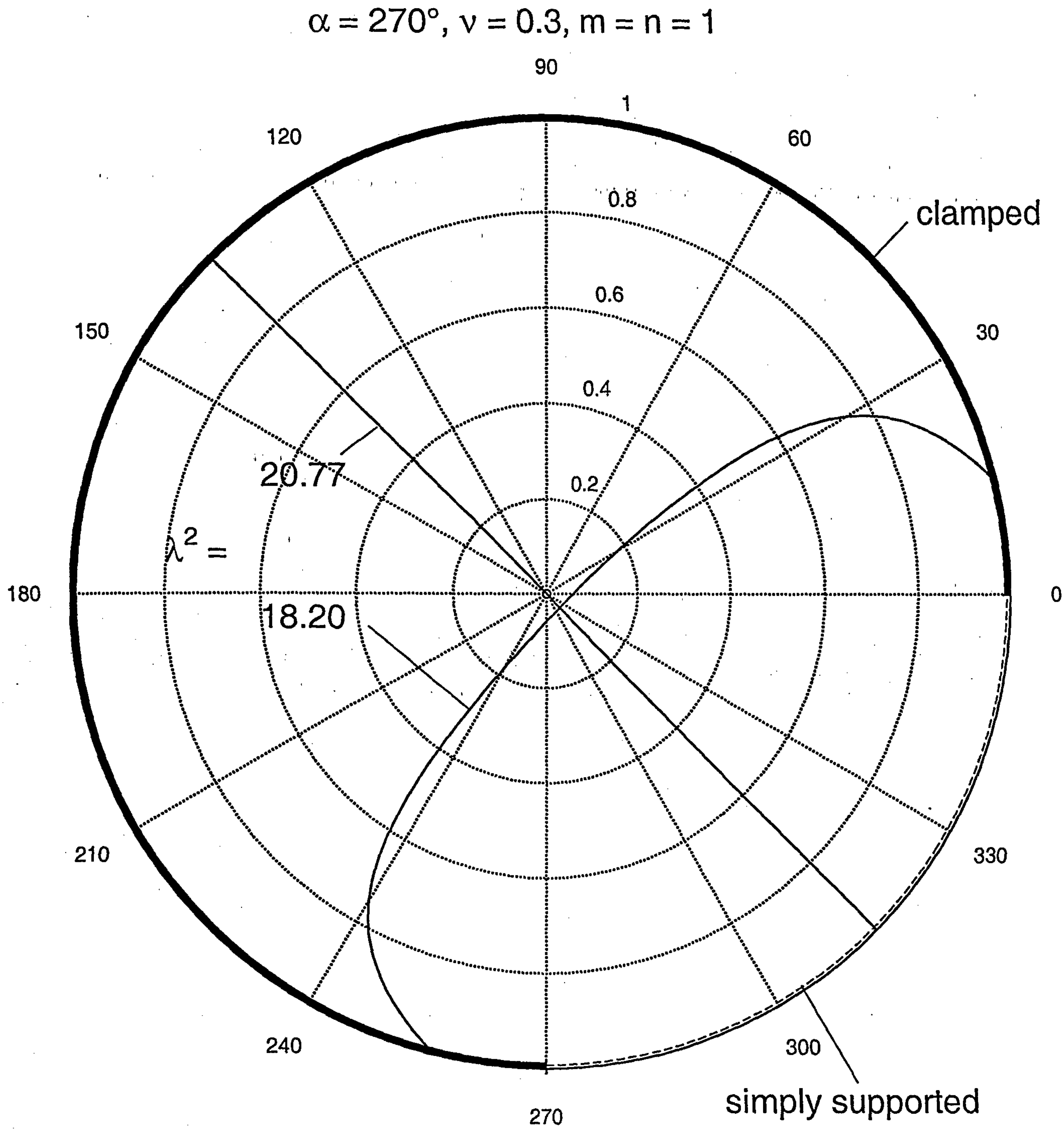

Fig. 5j H. F. Bauer/ W. Eidel

α = 300°, ν = 0.3, m = n = 1

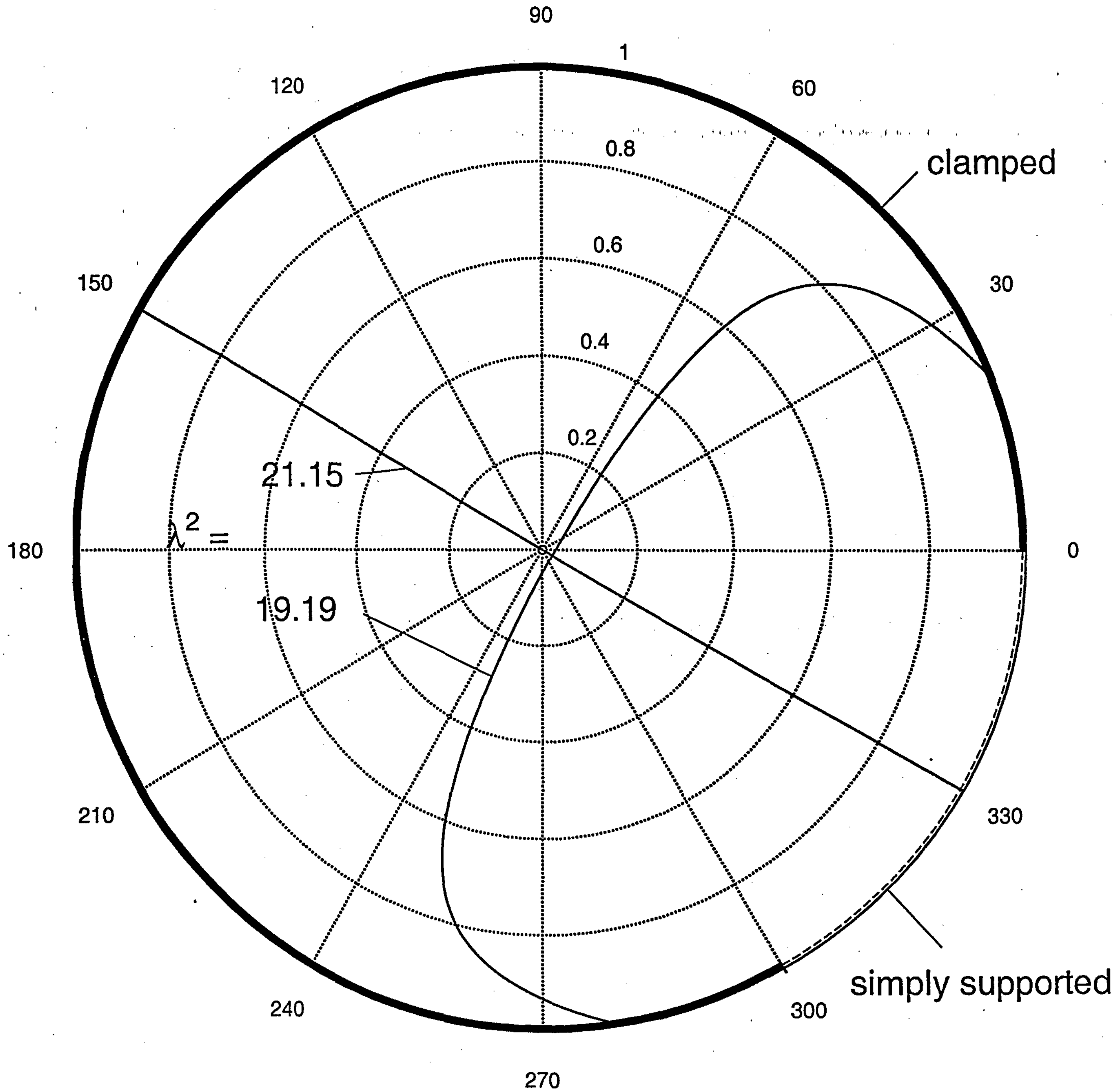

Fig. 5k H. F. Bauer/ W. Eidel

$\alpha = 350°, \nu = 0.3, m = n = 1$

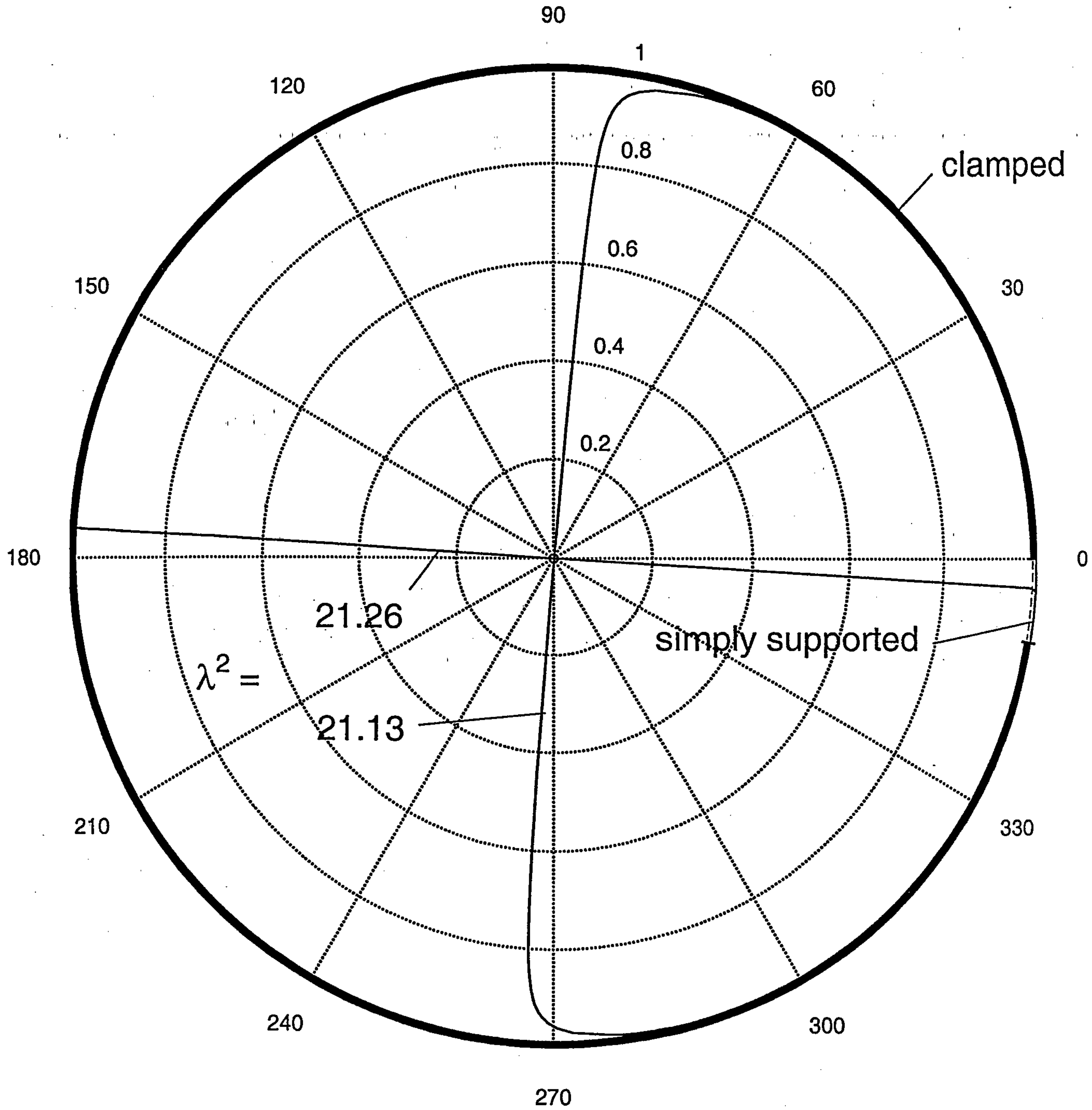

Fig. 5 *l* H. F. Bauer/ W. Eidel

$\alpha = 10°, \nu = 0.3, m = 2, n = 1$

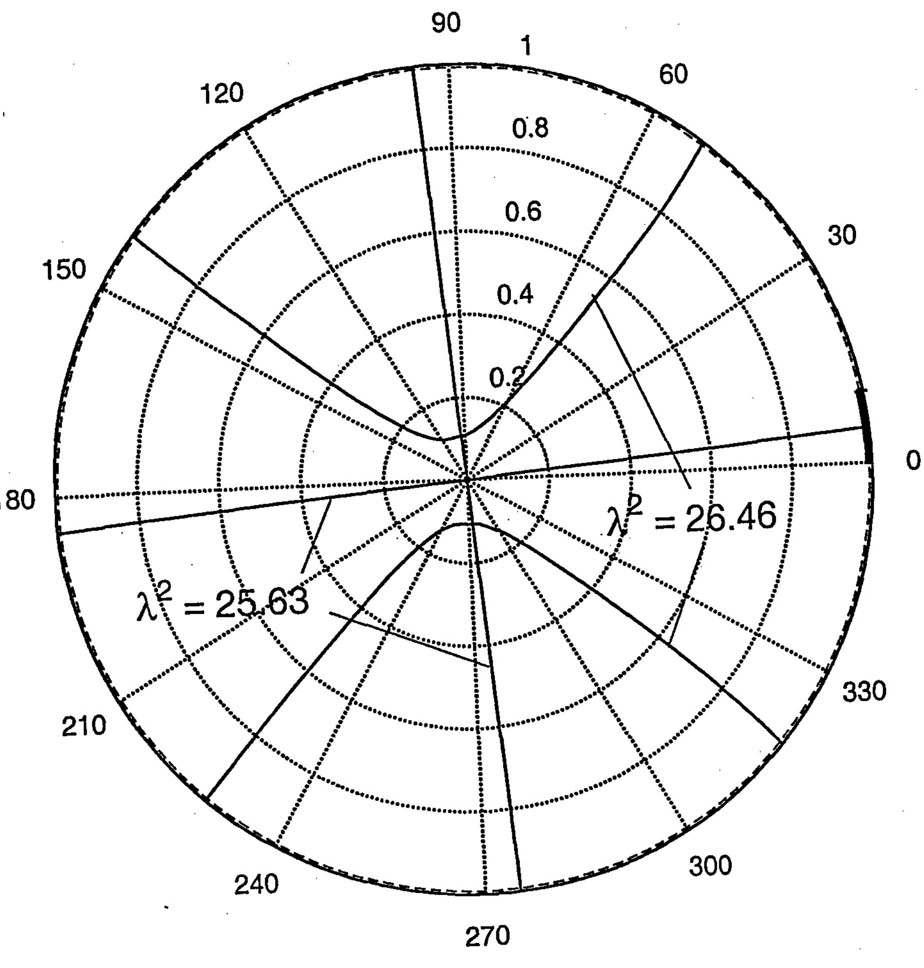

Fig. 6a H. F. Bauer/ W. Eidel

$\alpha = 30°, \nu = 0.3, m = 2, n = 1$

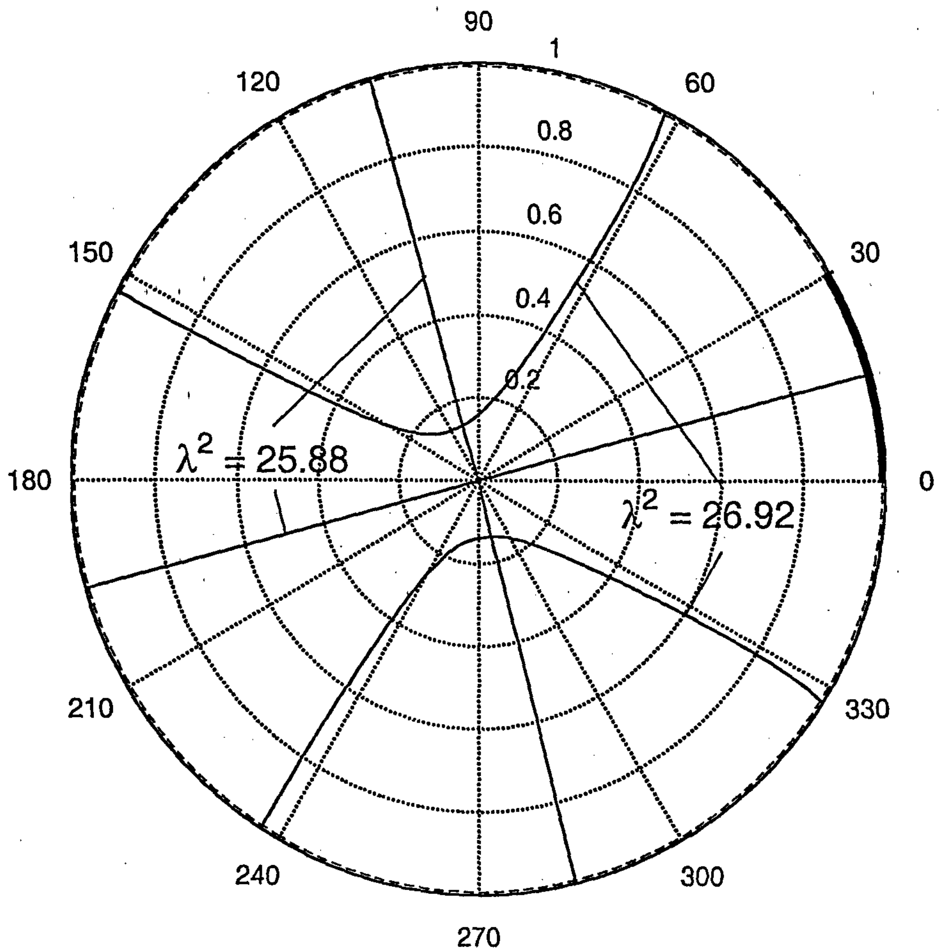

Fig. 6b H. F. Bauer/ W. Eidel

$\alpha = 120°, \nu = 0.3, m = 2, n = 1$

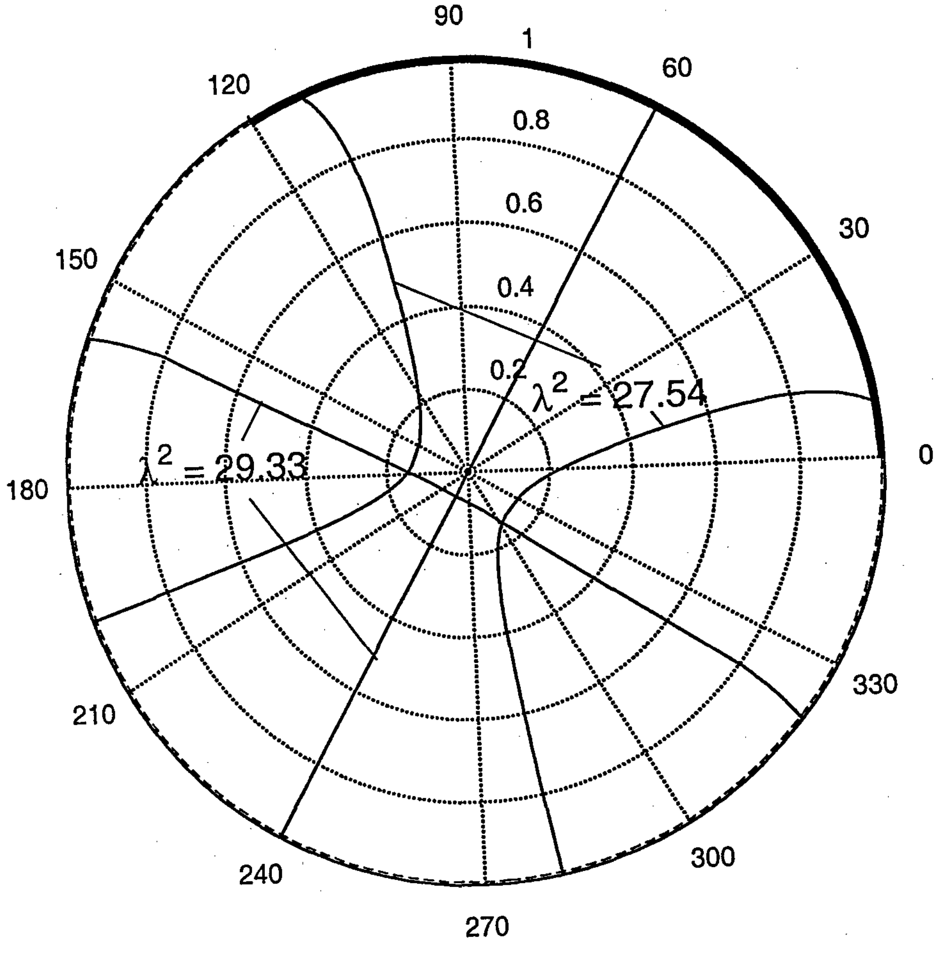

Fig. 6c H. F. Bauer/ W. Eidel

$\alpha = 170°, \nu = 0.3, m = 2, n = 1$

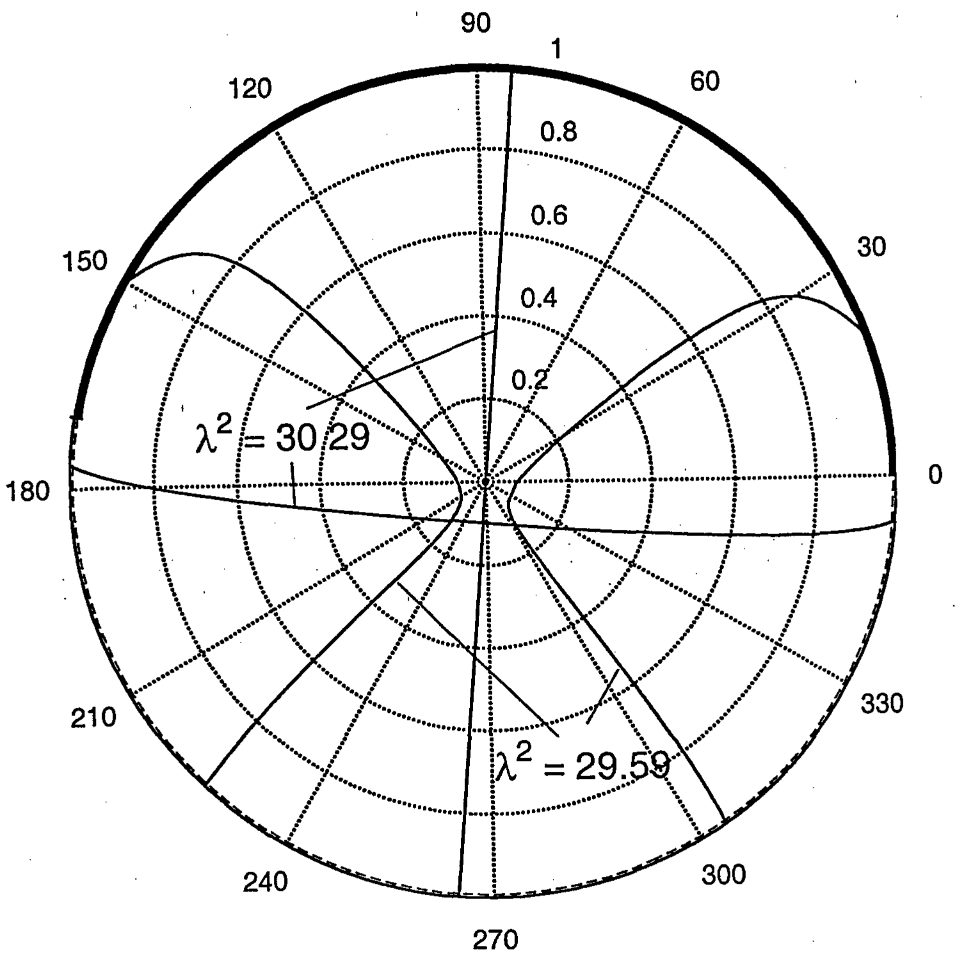

Fig. 6d H. F. Bauer/ W. Eidel

$\alpha = 190°, \nu = 0.3, m = 2, n = 1$

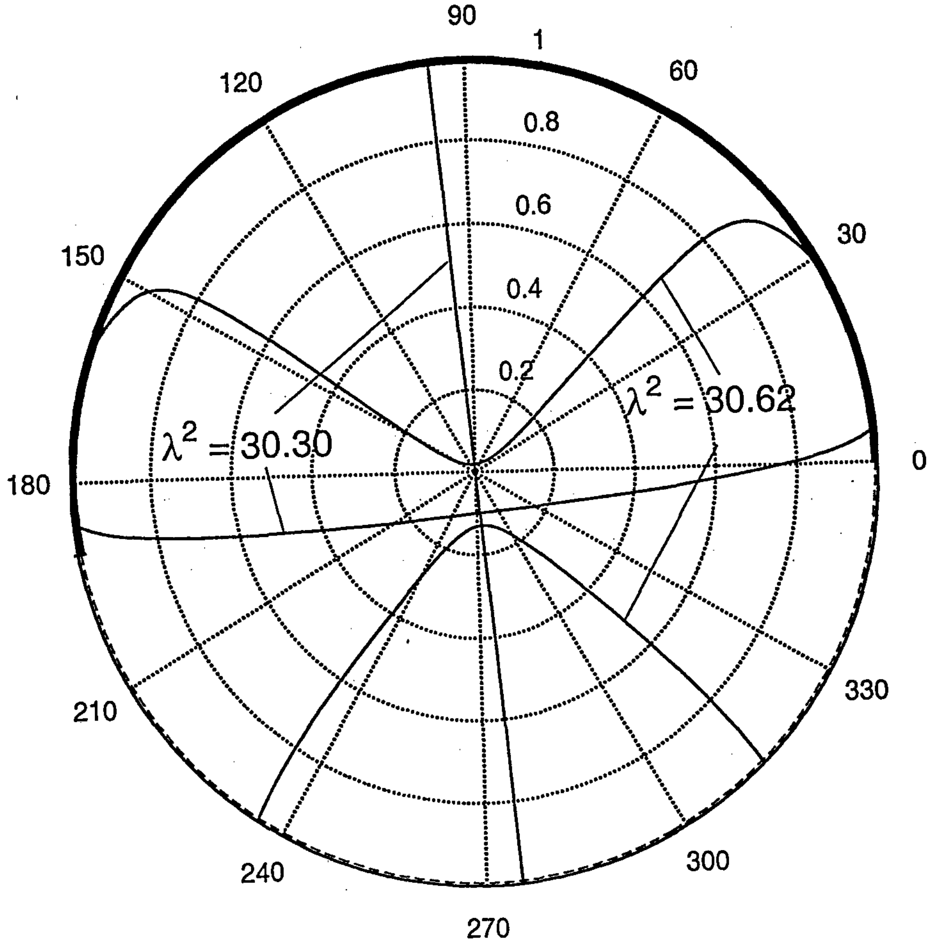

Fig. 6e H. F. Bauer/ W. Eidel

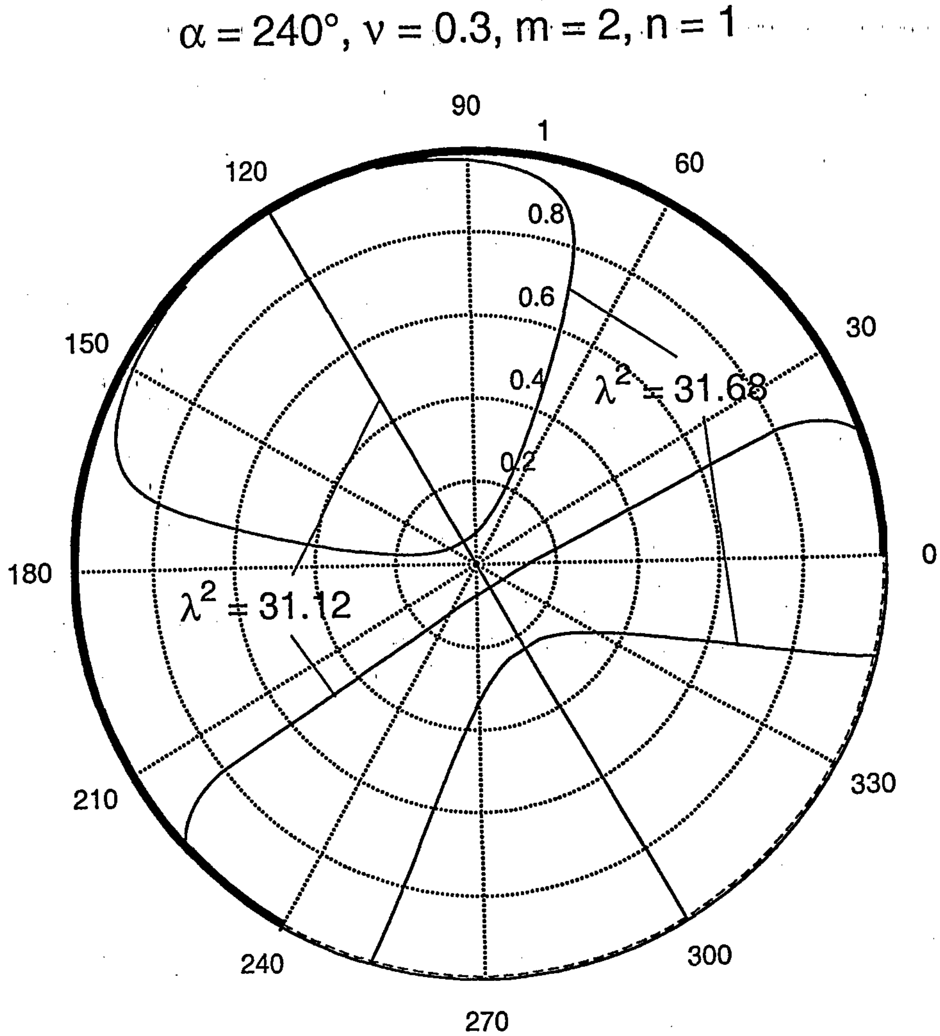

Fig. 6f H. F. Bauer/ W. Eidel

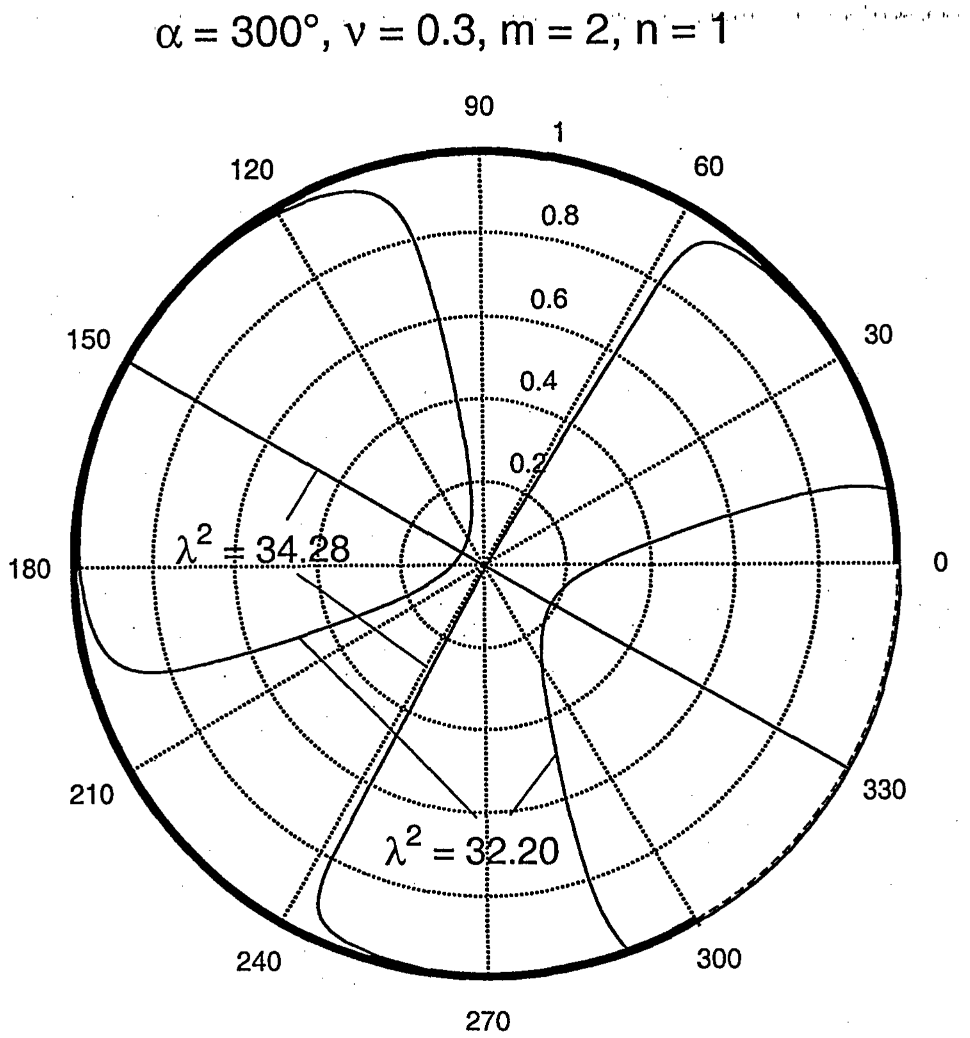

Fig. 6g H. F. Bauer/ W. Eidel